Random Matrix Theory, Quantum Chaos, and Eigenstate Thermalization

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In this paper, the main results from Random Matrix Theory (RMT), quantum chaos and Eigenstate Thermalization Hypothesis (ETH) will be surveyed. The theory behind quantum chaos and ETH is deeply rooted in RMT. It will be found that RMT can be used as an indicator for quantum chaos by analyzing the distribution of energy level spacing in quantum systems. ETH is a generalization of RMT in that one can reproduce RMT from ETH in a narrow energy window. The paper explores the connections in RMT, quantum chaos, and ETH.

I. INTRODUCTION

A classical system in which its motions are governed by non-linear equations can exhibit chaotic motions. Chaos in the classical sense means that the the system has an exquisite sensitivity to small perturbations in its phase-space trajectories. A famous example is the "Butterfly Effect", illustrating the point that any small difference in the initial condition can cause chaotic behavior.

On the other hand, this notion of classical chaos does not apply directly to a quantum mechanical system, in which its dynamics is governed by the linear Schrödinger equation. As in classical chaotic systems, one can use phase-space methods to describe quantum mechanical systems; however, the notion of trajectory is meaningless since the uncertainty principle forbids simultaneous measurement of coordinates and momenta. In such case, one may ask what chaos is, in the quantum mechanical setting.

Although, to this day, there is no precise definition of quantum chaos, it is referred to as chaos in the quantum mechanical setting. The basis for the understanding of quantum chaos has been formulated by Wigner and Dyson to understand the spectra of complex atomic nuclei, now called Random Matrix Theory (RMT).[1] The idea of this theory is that rather than finding the exact eigenspectra of chaotic Hamiltonians, one should instead probe their statistics.

Following this idea, Bohigas, Giannoni, and Schmit has formulated the BGS conjecture stating that quantum systems with a classical chaotic analog are described by RMT.[3] If the energy level spacing of arbitary quantum systems follow this he Wigner-Dyson distribution, then it can be said that the system is a quantum chaotic system. RMT acts as a tool to indicate whether a quantum system is chaotic or not.

RMT also has deep connections with thermalization of quantum mechanical systems. von Neumann's insight for quantum mechanical systems was that rather than focusing on their wave functions, one should instead focus on the observables. In his theorem, the quantum ergodic theorem, he stated that observable will relax to a microcanonical distribution at later times.[9] This relaxation of observables to the microcanonical distribution is called thermalization, a concept deeply connected to RMT. Since RMT was limited in describing physical observables in real systems, in the 1990's Srednicki generalized its notion by making an ansatz, known as the Eigenstate Thermalization Hypothesis (ETH).[1,4–6]

Following the discussions given by D'Alessio et.al.[1], this paper will survey some of the key concepts in RMT, quantum chaos and ETH. The goal of this paper is to illustrate how these concepts relate to one another.

II. CLASSICAL CHAOS

A. Phase space and Liouville's Theorem

Suppose we have a system of N point particle systems in which the position q_i and momenta p_i of each particle is known, where i indicates the i-th particle. For instance, let there be N particles. In this case, if the position and momentum for each particle have 3 components, then the subscript i will run from 1 to $3N$. A pair of position and momentum (q_i, p_i) is referred to as the degree of freedom. In the example above, the degree of freedom is then, $3N$. $3N$ of these pairs are needed to describe a dynamical system.

Consider a conserved system, or in other words, a system without any dissipation. A Hamiltonian of such system is described as a function of q_i and p_i that completely describes the system. The dynamics of the system can be expressed by considering the time evolution of the variables, q_i and p_i, namely, $q_i(t)$ and $p_i(t)$. In such case, the

following *Hamilton's equations*,[2]

$$\frac{\partial H(q,p)}{\partial p_i} = \frac{dq_i}{dt} \tag{1}$$

$$\frac{\partial H(q,p)}{\partial q_i} = -\frac{dp_i}{dt} \tag{2}$$

where $i = 1, \ldots, 3N$, if as in the above example, each particle has 3 components for their momenta and position. It is worth noting that the Hamilton's equations are coupled equations for each degree of freedom.

The Hamiltonian for the conserved system is dependent on momenta p, and position q. Let the system not be subject to time dependent forces. The total time derivative of the system can be found by the following,[2]

$$\frac{dH(q,p)}{dt} = \sum_i \left[\frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i}\frac{dq_i}{dt}\right] + \frac{\partial H}{\partial t}. \tag{3}$$

Due to the assumption that the system is not subject to time dependent forces, $\frac{\partial H}{\partial t} = 0$. Now, using the coupled differential equations for the position and momenta,[2]

$$\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} = 0. \tag{4}$$

If (4) holds true for each i in (3), it can be concluded that $\frac{dH(q,p)}{dt} = 0$ and H itself is a conserved quantity of the system. If H is the total energy, what ever the motion of the trajectories are in phase space (discussed in this section later), H is constant.

Next, the notion of *Phase space* of a conserved system will be mentioned. Phase space is a space created by position p and momenta q. Let a system under consideration have N degrees of freedom, then its dimensionality is $2N$ since each degree of freedom is represented by the pair q and p. In such conserved systems $H(q,p) = E$, where E is a constant energy value defined by the initial conditions of q and p. The trajectory that the system will follow in phase space will be confined to this $2N-1$ dimensional constant energy surface.[2]

Notice that there are multiple possibilities of initial conditions that the system can take. Therefore, we can define a distribution of initial conditions in phase space. Let the probability density function of the distribution be represented as $\rho(q,p)$. In this case, this function is defined so that the probability $P(q,p)$ of finding a trajectory in an infinitesimal volume dV in phase space is defined by $P(q,p) = \rho(q,p)dV$. The probability density function $\rho(p,q)$ can be shown to follow the following dynamics as the system evolves over time,[2]

$$\frac{d\rho}{dt} = \sum_i \left[\frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i}\frac{\partial H}{\partial q_i}\right] + \frac{\partial \rho}{\partial t}. \tag{5}$$

It can be shown that for a conserved system in which its Hamiltonian changes with time, the number of initial conditions inside a volume in phase space will stay constant. This allows us to make the notion that $\frac{d\rho}{dt} = 0$.[2] Applying this to (5), we get the following,[2]

$$\frac{\partial \rho}{\partial t} + \sum_i \left[\frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i}\frac{\partial H}{\partial q_i}\right] = 0. \tag{6}$$

This is called the *Liouville's theorem*. The terms inside the brackets describe how the density ρ varies with respect to q and p. On the other hand, $\frac{\partial \rho}{\partial t}$ describes the dynamics of ρ at a fixed point defined in phase space. What this theorem implies is that ρ does not evolve chaotically because it is linear in ρ. In other words, if the system starts in a different initial condition, then the probability distribution of the initial conditions will not evolve exponentially compared to the original distribution. This notion can be extended not only for non-chaotic systems, but also for chaotic systems in which the trajectories are diverging exponentially.[2]

B. Integrable and nonintegrable systems

Consider a conserved system with its trajectory in phase space start at initial condition (q_0, p_0). Since the system is conserved, the energy of the system will have the same value at later times, which leads us to write the Hamiltonian as $H(q(t), p(t)) = H(q_0, p_0)$. Here, $(q(t), p(t))$ is the trajectory in phase space. Recall Hamilton's equation considering momenta in (2). Consider a case where $\frac{\partial H}{\partial q_i} = -\frac{dp_i}{dt} = 0$. From this, we can say that when H does not have dependence on q_i, there is no time dependence on p_i. Therefore, in this system, the trajectories are constrained by the constant energy value $H(q_0, p_0)$ and p_i values. If the system has N degrees of freedom, and a total of M conserved quantities, then the trajectories will be on a $2N - M$ dimensional surface. There are two types of conserved systems: *integrable* and *nonintegrable* systems. A system is said to be integrable if the number of conserved quantities is equal or greater than the number of degrees of freedom. On the other hand, a system is considered nonintegrable if the number of conserved quantities is fewer than the number of degrees of freedom. In an nonintegrable system there are less constraints on the trajectories, therefore making them able able to travel through phase space more freely. In such system, chaotic motions are possible. In an integrable system, on the other hand, the motions of the system are constrained by the conserved quantities; therefore, in such a system, generally, the system will not show chaotic motion. However, there are cases in which integrable systems display sensitivity to initial conditions leading to chaotic behavior.

In this section, the phase space formulation of conserved classical systems have been discussed. In such formulation, it is possible to analyze chaotic behavior and classically, it is well understood. One may ask if there is a quantum analogy to these chaotic behaviors, namely

quantum chaos. This will be discussed in the following sections.

III. RANDOM MATRIX THEORY AND QUANTUM CHAOS

As described in the previous section, a system is called chaotic when it has an exquisite sensitivity to its initial conditions in its phase-space trajectories. One cannot directly use the phase-space trajectory argument in quantum mechanical systems since it is forbidden by the uncertainty principle to measure both coordinate and momenta simultaneously, making it impossible to define what a trajectory in phase space actually is. The notion of trajectory becomes meaningless in the quantum mechanical setting. Recall Hamilton's equations in (1) and (2). For chaotic motions to occur in classical mechanics, the Hamiltonian H must be nonlinear. In quantum mechanics, the Hamiltonian H is described by the Schrödinger's equation, which is linear. It is then obvious that one cannot smoothly apply the concepts in classical chaos directly to chaos in quantum mechanical systems. As mentioned in the introduction, chaos in quantum mechanical systems, or quantum chaos has its core theory deeply rooted in Random Matrix Theory (RMT). RMT is also the basis for eigenstate thermalization. This section will review RMT and its connections to quantum chaos and necessary concepts to understand the key concepts in eigenstate thermalization later developed in this paper.

A. Random Matrix Ensembles and Wigner-Dyson Distributions

Random Matrix Theory (RMT) is a theory to provide insight into the many properties of random matrices in which its entries are chosen randomly from specific probability distributions called random matrix ensembles. In order to illustrate the characteristics of RMT, first, consider a simple 2×2 Hamiltonian whose entries are taken from a Gaussian distribution with a mean of zero and with variance σ.[1,7]

$$\hat{H} = \begin{bmatrix} \lambda_1 & \frac{G}{\sqrt{2}} \\ \frac{G^*}{\sqrt{2}} & \lambda_2 \end{bmatrix} \tag{7}$$

The eigenvalues of the Hamiltonian can be found using the general procedure, and it is found that eigenvalues E_1, E_2 are

$$E_{1,2} = \frac{\lambda_1 + \lambda_2}{2} \pm \frac{\sqrt{(\lambda_1 - \lambda_2)^2 + 2|G|^2}}{2}. \tag{8}$$

The plot of E_1, E_2 with respect to $\lambda_1 - \lambda_2$ is given in Fig. 1. It is shown that E_1 and E_2 do not cross each other, or in other words, the energy levels are repulsive to each other. This is called energy repulsion.

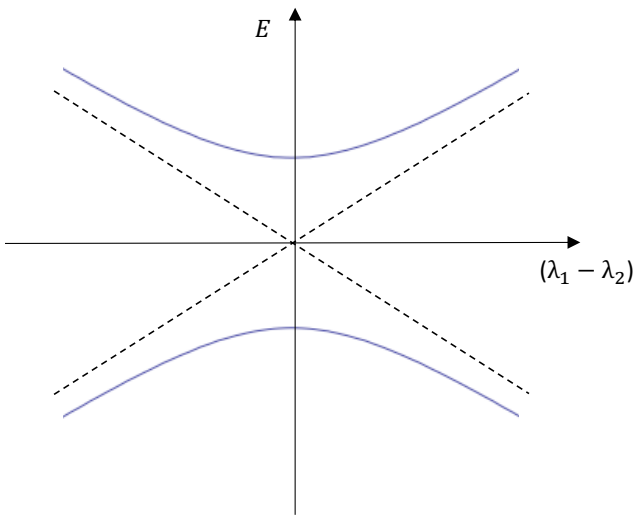

FIG. 1: Schematic of energy level repulsion in a 2×2 Hamiltonian.

If the two level system has time reversal symmetry, then $\hat{H}$ is a real matrix and hence its off diagonal entries satisfy $G = G^*$ by hermiticity. In the case where the system has its time reversal symmetry is broken, then the real and the imaginary values of the off diagonal entry G can themselves be treated as random variables, each chosen from Gaussian distribution. Let the level separation be expressed as $\omega = E_1 - E_2$. The probability distribution of the level separation $P(\omega)$ in the case when the system has time reversal symmetry, and in the case when it does not, are given in the following respectively.[1]

$$P(\omega) = \frac{\omega}{2\sigma^2} \exp\left[-\frac{\omega^2}{4\sigma^2}\right] \tag{9}$$

$$P(\omega) = \frac{\omega^2}{2\sqrt{\pi}(\sigma^2)^{3/2}} \exp\left[-\frac{\omega^2}{4\sigma^2}\right] \tag{10}$$

There are two characteristics that can be drawn from the statistics. Firstly, in the limit $\omega \to 0$, it can be seen that $P(\omega) \to 0$, indicating that there is a level repulsion between the two eigenvalues of the Hamiltonian E_1, E_2, as mentioned previously and shown schematically in Fig. 1. Secondly, $P(\omega)$ decays in the limit where ω is large. If we generalize the above distributions,[1]

$$P(\omega) = N_\gamma \omega^\gamma \exp\{[-M_\gamma \omega^2]\} \tag{11}$$

where γ is the variance of the distribution and an indicator of whether the system has time reversal symmetry. $\gamma = 1$ is when the system is time reversal symmetric and $\gamma = 2$ is when time reversal symmetry is broken. N_γ, M_γ are normalization constants from $P(\omega)$. This distribution in (11) is called the *Wigner surmise*. In particular, the energy level spacing distribution presented here for $\gamma = 1$ and $\gamma = 2$ are what the Gaussian orthogonal ensemble (GOE) and the Gaussian Unitary Ensemble

(GUE) follow respectively. The GOE and the GUE will be explained below.

Suppose there is an ensemble of random matrices of arbitrary size in which the components are drawn from a Gaussian distribution as in the 2×2 Hamiltonian example. The distribution of these random matrices are expressed as the following,[1,7]

$$P(\hat{H}) \propto \exp\left[-\frac{\gamma}{2k^2}\sum_{ij} H_{ij}H_{ji}^*\right], \tag{12}$$

where k represents the energy scale. There are three ensembles of matrices that can be extracted from this distribution. (i) *Gaussian orthogonal ensemble (GOE)* The ensemble of Hamiltonians with time reversal symmetry, $\gamma = 1$, and its components satisfy $H_{ij} = H_{ji}$. (ii) *Gaussian unitary ensemble (GUE)* The Hamiltonians do not have time reversal symmetry, $\gamma = 2$, and components satisfy $H_{ij} = H_{ji}^*$ due to the fact that Hamiltonian entries are complex. (iii) *Gaussian Symplectic Ensemble (GSE)* The ensemble of Hamiltonians that are invariant with respect to symplectic transformations. In this case, $\gamma = 4$.[7]

Note that these are the distribution for the random ensembles, not their energy level spacing as was shown for 2×2 Hamiltonian. When analyzing a certain complex quantum system, one will generate an ensemble of matrices such as the GOE and GUE that will represent the Hamiltonian of the system. In the general case of ensembles of larger matrices, unlike $P(\omega)$ in the 2×2 Hamiltonian system as in the example, the analytic form is hard to find, but its form is close to the Wigner surmise (11).[1,7] Such distribution of level spacing is called the *Wigner-Dyson distribution.* Note that "level spacing" here refers to the energy difference between neighboring energy eigenvalues.

As previously stated in the introduction, Bohigas, Giannoni, and Schmit discovered that the level spacing distribution of a single particle system followed the Wigner-Dyson distribution in a narrow window of high energies.[3] On the same paper, they made a conjecture that in the limit of high energies, if a classical system is highly chaotic, then it will have corresponding quantum energy levels that follow the Wigner-Dyson distribution.[3,10] If the level spacing distribution follows the Wigner-Dyson distribution, then it can be said that it is reflecting chaos in the quantum level, or quantum chaos. Studying these level spacing statistics from an ensemble of random matrices such as the GOE and the GUE lies at the core of RMT.

It is important to note the difference between a physical Hamiltonian and the Hamiltonian drawn from RMT. Most physical Hamiltonians are expressed as sparse matrices that are local, which can be non-random. On the other hand, the Hamiltonian drawn from RMT is a dense random matrix. It seems RMT cannot be applied to physical Hamiltonians; however, under the special condition that the physical Hamiltonian is non-integrable, its eigenspectra displays RMT statistics, the Wigner-Dyson distribution.

B. Matrix elements of observables

For the convenience of later discussions, let an operator $\hat{O}$ be called the observable of a given system. Since it is an observable, it can be thought of as a Hermitian operator, which can be represented as,[1]

$$\hat{O} = \sum_j O_j \left|j\right\rangle \left\langle j\right| \tag{13}$$

and that $\hat{O}\left|j\right\rangle = O_j \left|j\right\rangle$. Suppose a system can be represented as a Hamiltonian that is a true random matrix and that its eigenstates are represented as $|n\rangle$ and $|m\rangle$. It can then be said that the entries of the observable under the these eigenstates are,[1]

$$O_{mn} = \left\langle m\right| \hat{O} \left|n\right\rangle = \sum_j O_j (\Psi_j^m)^* \Psi_j^n \tag{14}$$

where $\Psi_j^m \equiv \langle j|m\rangle$ and $\Psi_j^n \equiv \langle j|n\rangle$ are the inner products of the eigenstates of the observable and the Hamiltonian. Before proceeding, it is necessary to briefly mention about the eigenvectors of the random matrices. The distribution of the components of the eigenvectors corresponding to the GOE and GUE can be written as $P_{GOE}(\Psi_1, \Psi_2, \ldots) \propto \delta\left(\sum_i \Psi_i^2 - 1\right)$, $P_{GUE}(\Psi_1, \Psi_2, \ldots) \propto \delta\left(\sum_i |\Psi_i|^2 - 1\right)$ respectively.[1,7] The probability distribution of the eigenvector components imply that eigenvectors of random matrices are essentially random unit vectors, that is, real or complex corresponding to GOE or GUE.[1] Now going back to the original discussion, if we let d represent the dimension of the Hilbert space in which it is spanned by the eigenstates of the Hamiltonian, then the average of the product between eigenvector components over random eigenstates can be expressed as $\overline{\Psi_i^m \Psi_j^n} = \frac{1}{d}\delta_{mn}\delta_{ij}$.[1] This is due to the orthogonality between the eigenvectors that are essentially random unit vectors as previously mentioned. Using this relation, the average value of the observable components can be represented as the following[1],

$$\overline{O_{mn}} = \begin{cases} \frac{1}{d}\sum_i O_i, & \text{for } m = n \\ 0, & \text{for } m \neq n \end{cases} \tag{15}$$

This shows that when evaluating for the average value of the observable components one finds that only the diagonal entries are nonzero, and that it gets suppressed by the dimension of the Hilbert space.

The fluctuations of the observable entries can be shown to satisfy the following,[1]

$$\overline{O_{mn}^2} - \overline{O_{mn}}^2 = \begin{cases} \frac{3-\gamma}{d^2}\sum_i O_i^2, & \text{for } m = n \\ \frac{1}{d^2}\sum_i O_i^2, & \text{for } m \neq n \end{cases} \tag{16}$$

Assuming that the fluctuations are small and independent with respect to the dimension of the Hilbert space, (16) tells us that the fluctuation gets smaller and smaller with increasing dimension. Using these expression for the fluctuations, one can express the observable entries as the following to the leading order $1/d$,[1]

$$O_{mn} \approx \bar{O}\delta_{mn} + \sqrt{\frac{\overline{O^2}}{d}} R_{mn} \tag{17}$$

where it was defined that $\overline{O} \equiv \frac{1}{d}\sum_j O_j$, $\overline{O^2} \equiv \frac{1}{d}\sum_j O_j^2$ and that R_{mn} is a component of a random matrix chosen from a Gaussian distribution of zero mean and the variance of either $\gamma = 1$ for (GOE) and $\gamma = 2$ for (GUE), in which it will be real and complex respectively. The average expressions given in these set of equations are averaged over the eigenstates of each of the Hamiltonians in an ensemble. As long as the dimension of the Hilbert space d is large enough, or in other words, the samples of the Hamiltonians are large enough, the observable of a system governed by a fixed Hamiltonian can be approximated using equation (17).

C. Quantum Integrability

In classical integrable and nonintegrable systems, the notion of chaos is well defined via trajectories in phase space. However in the quantum setting, the notion of chaos is nontrivial. To understand quantum chaos, one can use RMT and look at many different statistical features to analyze the random matrix eigenspectrum; however, the common procedure is to probe the distribution of energy level spacing of the eigenstates. In the previous section, it was stated if the level spacing distributions of the quantum system follows the Wigner-Dyson distribution, then it is an indication of quantum chaos. As in classical systems, one may ask if there exists integrability in the quantum setting and if there exists its indicator for it. In the quantum setting, integrals of motion and its indicator does exist. As the Wigner-Dyson distribution was the distribution of energy level spacing for quantum chaotic systems and the indicator for such systems, the distribution of energy level spacings for quantum integrable systems follow a Poisson distribution,[1]

$$P(\omega) = \exp[-\omega]. \tag{18}$$

where ω is the energy level spacing and the mean level spacing was set to unity. The main difference of this distribution from the Wigner-Dyson distribution, which again follows a form similar to the Wigner surmise in (11), is that the Poisson form does not have energy level repulsion. From this view point, it can be said that to probe the distribution of energy level spacing is to probe the energy level repulsion.

Recall Liouville's theorem in (6). If the conserved system started with a slightly different probability density ρ, then the new probability distribution will not exponentially diverge from the original probability distribution, even if the system itself is chaotic or not.[2] The role of the probability distribution in phase space in the quantum setting is played by the Wigner-Dyson and Poisson distributions.

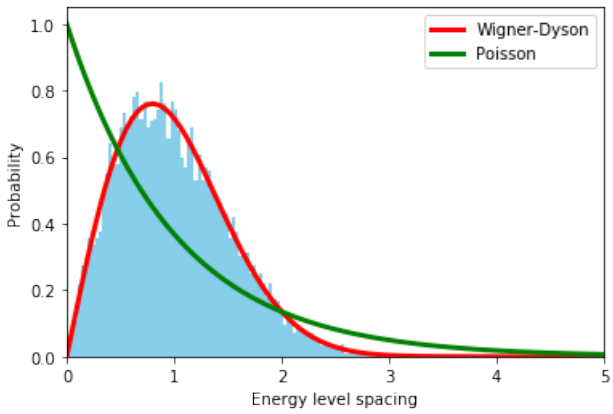

FIG. 2: GOE following Wigner-Dyson distribution. The Poisson distribution is plotted for convenience.

To illustrate the above discussions, suppose we generate an ensemble of random matrices. To be more specific, let the ensemble be GOE. In Fig.2, the energy level spacings for the GOE are shown as the histogram. It can be seen that the energy level spacings follow the Wigner-Dyson distribution. This indicates that the system shows signs of quantum chaos. For convenience, the Poisson distribution was plotted together in Fig. 2. In the case where the random matrix ensemble follows the Poisson distribution, the system is said to be integrable.

To summarize, if one finds that the level spacing distribution follows Poisson or Wigner-Dyson distribution, then it means that they have found a signature for a quantum integrable or chaotic system respectively.

IV. EIGENSTATE THERMALIZATION

A. Classical Thermalization

Let O represent an observable of an isolated system. According to equilibrium statistical mechanics, the time average of such observables over some time interval T, $\langle O \rangle_T$ are often what gets quantified. This time interval T will be extended to infinity to account for the system to reach thermal equilibirum, or thermalization. However, when the time interval is taken to infinity, it seems impossible to keep track of the value of the observables for all times. It is necessary to find a different way to calculate this quantity. Recall the phase space formulation in classical mechanics as introduced in section II of this paper. If a system get arbitrarily close to following every possible point in phase space in its phase space trajectory, $\Lambda(t)$,

then the system is defined to be *ergodic*.[10] In the limit of long times, ergodicity states that the phase space trajectory $\Lambda(t)$ will follow every point in a constant energy surface.[10] Furthermore, Liouville's theorem implies that this phase space trajectory will end up covering the constant energy surface uniformly.[10] This leads to the fact that the time average of an observable $\langle O \rangle_T$ can also be calculated by averaging over the phase space, which is confined by a constant energy surface ES; therefore,[10]

$$\langle O \rangle_T = \frac{\int_{ES} O(\Lambda) d\Lambda}{\int_{ES} d\Lambda}. \tag{19}$$

This is the formulation of averages in the microcanonical ensemble in the classical setting. Classically, the fact that the long time average is equal to the microcanonical average is well defined. This is not so in the quantum setting.

B. Quantum Thermalization

Quantum statistical mechanics postulates that a quantum system is in contact with an external reservoir.[11] When a quantum system is not in contact with such reservoir, it is called a closed quantum system. However, it is possible to divide such system into a subsystem, which contain a fraction of the total degrees of freedom in the whole system, and the remaining portion of the closed system as a reservoir.[11] If the rest of the system can act a reservoir for the small subsystem, then we can recover the notions of quantum statistical mechanics again and at late times, the subsystem can undergo thermalization. Before proceeding, note that observables in closed quantum systems are local. In a given system, we cannot track all the degrees of freedom upon measurement. Instead, we can only measure a small portion of the entire system, which what makes the observables very local.

Suppose there is a closed quantum system such that it is initially in the pure state $|\psi_0\rangle$ and its dynamics are described by a time independent Hamiltonian $\hat{H}$ that has energy eigenvectors $|m\rangle$ with eigenvalues E_m, $\hat{H}|m\rangle = E_m|m\rangle$.[1] If we want to find the time-evolving state of the system $|\psi(t)\rangle$, we use the corresponding Schödinger equation, which is expressed as $i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle = E_m|\psi(t)\rangle$. The time-evolving state of the closed quantum system can be expressed as the following,[1]

$$|\psi(t)\rangle = \sum_m C_m e^{\frac{-iE_m t}{\hbar}} |m\rangle \tag{20}$$

where $C_m = \langle m|\psi(0)\rangle = \langle m|\psi_0\rangle$. Now that we have an expression for the time evolution of the state of the closed quantum system, we can shift directions to analyzing the time averaged expectation value of the observable $O(t) \equiv \langle\psi(t)|\hat{O}|\psi(t)\rangle$. Using this time dependent evolution of the state vector, the time evolution of the observable $O(t) \equiv \langle\psi(t)|\hat{O}|\psi(t)\rangle$ can be expressed as

$$O(t) = \sum_m |C_m|^2 O_{mm} + \sum_{m\neq n} C_m^* C_n e^{\frac{i(E_m - E_n)t}{\hbar}} O_{mn} \tag{21}$$

where $O_{mn} = \langle m|\hat{O}|n\rangle$.[1] Here it can be seen that the time averaged expectation value $O(t)$ is expressed in terms of expectation values of these observables under energy eigenstates $|m\rangle$ and $|n\rangle$, O_{mm} and O_{mn}. If we take the long time average of $O(t)$, the second term in (21) will go to zero (due to exponential decay with system size) under the assumption that there are no degeneracies. On the other hand, the microcanonical average in the quantum setting involves averaging over all eigenstates within a small energy window ΔE.[10] For large systems, this energy window ΔE is very small, and this itself contains many eigenstates. The microcanonical average value is given by averaging over all the energy eigenstates in this window, which is centered around the mean energy of the initial state of the system.[12] The microcanonical average of the observable $\overline{O}$ is expressed as the following,[12]

$$\overline{O} = \frac{1}{N_{eigen}} \sum_m O_{mm} \tag{22}$$

where N_{eigen} is the number of eigenstates in the small energy window ΔE. In order for thermalization to occur for this closed quantum system, the microcanonical average $\overline{O}$ in (22) must equal the long time average value of (21), which is just the first term $\sum_m |C_m|^2 O_{mm}$. This equivalence is the challenge of defining thermalization in closed quantum systems. It is then natural to ask if there is a way to reconcile this problem.

If we draw the Hamiltonian $\hat{H}$ out of RMT, we can use the notion of observables under RMT (17). This states that the diagonal elements of the observable, O_{mm} is independent of energy eigenstate $|m\rangle$ and that the off-diagonal elements O_{mn} are exponentially small in system size.[1] Applying these statements from RMT, the first term in (21) will then become $\sum_m |C_m|^2 O_{mm} \approx \overline{O}\sum_m |C_m|^2 = \overline{O}$. The second term of (21) will vanish since the off-diagonal elements become uncorrelated. If we denote the long time average of $O(t)$ as $\langle O(t)\rangle_{LT}$, then the above arguments make $\langle O(t)\rangle_{LT} = \overline{O}$, which is essentially the long time average of the observable is equal to the microcanonical average, a statement of thermalization.

In practice, RMT is not enough to describe observables in physical systems. As we have seen above, we were only able to see the microcanonical average equal to the long time average of the observable when we assumed that the Hamiltonian $\hat{H}$ of the closed quantum system was drawn from RMT, meaning that the $\hat{H}$ matrix is a dense matrix. However, in a physical system, the Hamiltonian is sparse and local. A certain generalization of RMT is necessary to account for observables in real systems, which was done by Srednicki who made an ansatz known as the Eigenstate Thermalization Hypothesis (ETH).[1,4–6].

C. Eigenstate Thermalization Hypothesis (ETH)

Srednicki made the following ansatz that will generally hold for physical observables,[1,4–6]

$$O_{mn} = O(E)\delta_{mn} + e^{-S(E)/2} f(E,\omega) R_{mn} \tag{23}$$

where $E \equiv (E_m + E_n)/2, \omega \equiv E_n - E_m$ and $S(E)$ is the thermodynamic entropy at the argument energy E that is given by, $e^{S(E)} = E\sum_m \delta(E - E_m)$. The functions $O(E)$ and $f(E,\omega)$ are smooth functions with respect to their arguments and R_{mn} is a real or complex numerical factor that varies randomly with m and n such that $\overline{R_{mn}^2} = 1$ and $\overline{|R_{mn}|^2} = 1$. Note that the observable represented in this ansatz is constructed by taking the eigenstates of the Hamiltonian as the basis. If the Hamiltonian of the system obeys time-reversal symmetry, the eigenstate components are real hence the matrix components of the observables are real as well. If time reversal symmetry is broken, the matrix components of the observables are complex values. Without losing generality, we can take the function $f(E,\omega)$ to be an even function in the argument ω.[1] Taking the Hermitian conjugate of equation (14), the random numerics R_{mn} and the function $f(E,\omega)$, satisfies $R_{mn} = R_{nm}$ and $f(E,-\omega) = f(E,\omega)$ for a system with time reversal symmetry and $R_{mn} = R_{nm}^*$ and $f(E,-\omega)^* = f(E,\omega)$ for a system without time reversal symmetry.[1]

Comparing this ETH ansatz with the RMT prediction of the observables given in equation (17), the diagonal elements in the ansatz, $O(E)$, are no longer same values, but smooth functions in their arguments E and ω. On the other hand, the off-diagonal elements contain a thermodynamic entropy factor to account for the thermal fluctuations in physical systems and a smooth function $f(E,\omega)$. As stated previously, the ansatz is also a generalization of the RMT prediction of observables in (17). The ETH ansatz reduces to the RMT prediction in a narrow energy window called the Thouless energy E_T, a measure of particles' sensitivity to feel the boundaries of the system expressed in terms of energy.[8] In fact, E_T is the smallest possible energy window. It is defined as $E_T = \frac{\hbar D}{L^2}$ where L is the edge length of the system and D is the diffusion constant. If the energy level separation ω satisfies $\omega < E_T$, then the function $f(E,\omega)$ is constant in that region because it is smooth in its arguments. Giving such restriction to ω makes the ETH ansatz approximately equal to RMT prediction for the observables shown in equation (17). Note that this reduction to ETH to RMT occurs only for diffusive systems with diffusion time t longer than L^2. Diffusion is in fact a result of physical systems with local interactions leading to local Hamiltonians. Once $\omega < E_T$ is not satisfied, the ETH ansatz will no longer appear as a RMT prediction. From the argument made here, it seems RMT is useless because of its restrictive necessary condition. Notice however, that as the system size increases, energy level spacings become closer together. Even in the restriction that $\omega < E_T$, there are exponentially many energy levels where RMT applies. In this way, RMT and ETH are related to each other with latter being the generalization of the former.

D. ETH and Thermalization

As stated in the introduction, von Neumann stated in his theorem, the quantum ergodic theorem, that a macroscopic observable will relax to a microcanonical distribution at later times.[1] In this section, by following this theorem, it will be explicitly shown how the concept of thermalization is related to ETH.

In this theorem,von Neumann considered a quantum system with N interacting particles confined in a finite volume. Due to the confinement of the system in a closed space, the Hamiltonian $\hat{H}$ governing the system will form discrete energy levels E_m such that $\hat{H}|m\rangle = E_m|m\rangle$ where $|m\rangle$ is the corresponding eigenstates or the orthonormal basis of the system's Hilbert space $\mathcal{H}$ of dimension d. The theorem focuses in the microcanonical energy window δE, so it can be expected that $E_m \in (E - \delta E/2, E + \delta E/2)$. In this window, we can define an arbitrary state ψ in terms of eigenstates $|m\rangle$ such that $|\psi\rangle = C_m|m\rangle$ where $C_m = \langle m|\psi\rangle$. The total Hilbert space $\mathcal{H}$ can be decomposed in terms of mutually orthogonal Hilbert subspaces such that $\mathcal{H} = \bigoplus_\nu \mathcal{H}_\nu$ and $d = \sum_\nu d_\nu$ where $\mathcal{H}_\nu$ refers to the Hilbert space of microstate ν and d_ν as its dimensionality. What von Neumann meant when he said "macroscopic" is that it is built upon microstates ν. The macroscopic observable $\hat{O} = \sum_\nu O_\nu \hat{P}_\nu$ where $\hat{P}_\nu$ is the projector to $\mathcal{H}_\nu$. Let $O(t)$ be the observable at time t such that $O(t) = \langle\psi| e^{\frac{i\hat{H}t}{\hbar}} \hat{O} e^{\frac{-i\hat{H}t}{\hbar}} |\psi\rangle$. The microcanonical average is given as $\langle O\rangle_{ME} = \sum_m \langle m|\hat{O}|m\rangle/d$. We then require for $\hat{H}$ to not have any nondegenerate energies, $E_m - E_n \neq E_{m'} - E_{n'}$ unless $m = m', n = n'$ or $m = n, m' = n'$. The central statement of the quantum ergodic theorem is of the following,[1,9]

$$F = \max_{m\neq n} |\langle m|P_\nu|m\rangle|^2 + \max_m \left(\langle m|P_\nu|m\rangle - \frac{d_\nu}{d}\right)^2. \tag{24}$$

If F is exponentially small, then $|O(t) - \langle\hat{O}\rangle_{ME}|^2 < \epsilon\langle\hat{O}^2\rangle_{ME}$ where δ is a small number. If we compute for the expectation value of the observable $\hat{O}$, we find the diagonal components as

$$\begin{aligned}\langle m|\hat{O}|m\rangle \approx \sum_\nu O_\nu \frac{d_\nu}{d} &= \sum_{m\in\mathcal{H},\nu} O_\nu \frac{\langle m|\hat{P}_\nu|m\rangle}{d} \\ &= \sum_{m\in\mathcal{H}} O_\nu \frac{\langle m|\hat{O}|m\rangle}{d} \\ &= \langle O\rangle_{ME}\end{aligned} \tag{25}$$

where $\sum_{m\in\mathcal{H}} \langle m|\hat{P}_\nu|m\rangle$.[1] This states that the diagonal components are equal to the microcanonical average

value, which is indeed what von Neumann stated in the theorem. For the off-diagonal components,on the other hand, we find,

$$\langle m|\,\hat{O}\,|n\rangle = \sum_{\nu} O_{\nu}\,\langle m|\,\hat{P}_{\nu}\,|m\rangle\,. \tag{26}$$

However from (24), it is required that $\langle m|\,\hat{P}_{\nu}\,|m\rangle$ is exponentially small and O_{ν} is a finite physical observable; therefore, the off-diagonal components are exponentially small in system size.[1] Note that $O_{mm} = \langle m|\,\hat{O}\,|m\rangle$ and $O_{mn} = \langle m|\,\hat{O}\,|n\rangle$. What is significant about this results is that this approximately equal to the RMT prediction for observables presented in (17), which in other words, ETH in Thouless energy window $\delta E \sim E_T$.

V. CONCLUSION

In this paper, the main results from RMT, quantum chaos, and ETH has been surveyed. One can say that RMT is deeply rooted in the theory behind quantum chaos and ETH. Using RMT, one is allowed to find the distributions for energy level spacings. If the the energy level spacing of a quantum system follows the Wigner-Dyson distribution, then the system is said to be a quantum chaotic system. If on other hand, the spacing follow Poisson distribution, then the system is said to be quantum integrable system. Therefore, RMT can be used as an indicator for whether the system is chaotic or not. As in quantum chaos, ETH is also deeply related to RMT. ETH was formulated to understand thermalization of observables in physical systems. ETH is actually a generalization of RMT in a way that ETH reduces to the RMT prediction of observables when considering energy level spacing within the Thouless energy window. It is remarkable how powerful RMT is in establishing methods to probe quantum chaos and acting as a basis for ETH.

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