Stability of MBL in two dimensions and higher

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Recent discovery of many-body localization(MBL) in disordered systems have sparked intense excitement. Unlike typical non-integrable many-body quantum systems, MBL systems do not equilibrate under intrinsic dynamics, even at energy densities corresponding to infinite effective temperatures. In one spatial dimension, the existence of a stable MBL phase has been essentially confirmed, but the stability of MBL in higher dimensions remain controversial. This paper examines the "avalanche" argument against the existence of MBL in two dimension and higher, discusses its possible failure mechanisms, and reviews its implications for future theory and experiment efforts.

I. INTRODUCTION

A non-integrable many-body quantum system evolving under its internal dynamics will, in general, thermalize at long times to some thermal ensemble set by the initial energy density of the system. On the other hand, many-body localized (MBL) systems do not equilibrate under intrinsic dynamics, even at energy densities corresponding to high or infinite effective temperatures. Currently, MBL is the only known generic, stable exception to thermalization. Evading thermalization allows us to preserve interesting quantum effects at high temperatures, and is hence desirable for a wide range of applications. In one spatial dimension, the existence of a stable MBL phase has been essentially confirmed by theory, numerics, and experiment. However, it is not clear if stable MBL phases can exist in dimensions two and higher. There are theoretical arguments for the instability of MBL[1], but there also exists experimental observation of a MBL-like phase in a two dimensional system[2]. In this report, I would like to first introduce relevant definitions of thermalization and MBL. Then, I go over the theoretical arguments against the existence of MBL in two dimensions and higher[1], which involves rare thermal inclusions driving an "avalanche" that thermalizes the entire system. Finally, I discuss the implications of these arguments, including how we can reconcile them with experiments like[2], as well as and other possible routes for evading thermalization in higher dimensions. In writing this report, I have found the reviews[3–5] to be very helpful.

II. RELEVANT BACKGROUND

The conventional framework of condensed matter physics concentrates on studying phases of matter and phase transitions in thermodynamic equilibrium. We imagine the system we wish to study to be in contact with an external reservoir. In this setting, we have well-defined notions of temperature and thermal emsembles, and states of the system can be specified by a few conserved quantities, such as energy and particle number. From a partition function, which may look something like

$$\mathcal{Z} = \mathrm{tr}\left(e^{-\beta H}\right), \tag{1}$$

we can calculate any macroscopic thermodynamic observables we desire, such as average energy, pressure, specific heat, and entropy.

Initially, it was unclear why and when we can use the above framework to describe isolated quantum systems, which do not exchange energy and particles with the environment and evolve under unitary dynamics. The eigenstate thermalization hypothesis (ETH)[6] provides sufficient conditions for isolated quantum systems to thermalize. More explicitly, the ETH states that if an observable O, written in some eigenbasis $\{|m\rangle\}$ of the Hamiltonian, has the form

$$\langle m|O|n\rangle = O_{\mathrm{MC}}\left(\bar{E}\right)\delta_{mn} + e^{-S(\bar{E})/2} f_O(\bar{E},\omega)\eta_{mn} \tag{2}$$

where

$$\bar{E} = \frac{E_m + E_n}{2}, \quad \omega = E_n - E_m, \tag{3}$$

$S(\bar{E})$ is the entropy at energy $\bar{E}$, $O_{\mathrm{MC}}(E)$ is the expectation value of O in the microcanonical ensemble with energy E, η_{mn} is a random variable with zero mean and unit variance, and $O_{\mathrm{MC}}(E)$ and $f_O(E,\omega)$ are both smooth functions of their arguments, then the observable O thermalizes in the sense that

1. The long time average O_T agrees with the thermal value $[\rho_{\mathrm{th}} O]$ for an appropriately chosen thermal ensemble ρ_{th}

$$O_T = \lim_{T\to\infty}\int_0^T \langle\psi(t)|O|\psi(t)\rangle = \mathrm{tr}\left[\rho_{\mathrm{th}} O\right] \tag{4}$$

2. Fluctuations about this long time average is small at most late times.

ETH specifies sufficient conditions for an isolated quantum system to act as reservoir for its subsystems, and bring itself to thermal equilibrium. Most few-body observables in non-integrable quantum systems obey ETH. We expect these systems to look, in a sense, "classical", because nontrivial quantum correlations in the initial state are hidden in inaccessible, global operators at long times.

Thermalization is not the only possible fate for isolated quantum systems. The other generic outcome is many-body localization, which occur in systems with strong disorder. Intuitively, the MBL phase occurs whenever the transition rate between two eigenstates close in energy is much smaller than the many-body level spacing. The MBL phase is defined by the existence of an complete set of commuting local integrals of motion (LIOMs) τ_i^z, which are connected to exactly local spin operators σ_i^z by a quasi-local unitary trasnformation U. Throughout the MBL phase, the system can be described by the effective hamiltonian

$$H_{\mathrm{MBL}} = \sum_i h_i \tau_i^z + \sum_{ij} J_{ij} \tau_i^z \tau_j^z + \sum_{ijk} J_{ijk} \tau_i^z \tau_j^z \tau_k^z + \ldots \quad (5)$$

where h_i are random site energies, and $J_{ij}, J_{ijk}, \ldots$ exponentially decay with distance between LIOMs. Many interesting and desirable properties of MBL, including local memory, area-law entanglement, and localization protected quantum order, can then be derived from this LIOM picture.

The non-interacting version of MBL, the so-called Anderson localization, or single-particle localization, has been extensively studied and is fairly well understood. Anderson's original paper shows that the Hamiltonian above realized on a $d = 3$ cubic lattice has a phase transition between a strongly localized, non-thermalizing phase and a delocalized, thermalizing phase[7]. The relevant parameter is the ratio of disorder strength to single-particle bandwidth. Subsequent scaling arguments have shown that in $d \leq 2$, even arbitrarily weak disorder is sufficient to produce a localized system[8]. Recent work also shows that stable Anderson localized phases exist in arbitrarily high dimensions[9]. By comparison, true many-body localization is only (almost) proven to exist in one dimension. Imbrie[10] explicitly constructed LIOMs for a particular class of Ising spin Hamiltonian, and numerical studies of the spin-1/2 random field Heisenberg model[11] points to a phase diagram which looks roughly like Fig. 1.

Comparing Anderson localization and MBL in Table I, it is clear that interactions strongly destabilize the localized phase. It is entirely conceivable (albeit disappointing) that thermalization is the only possible generic outcome for interacting quantum systems in $d \geq 2$.

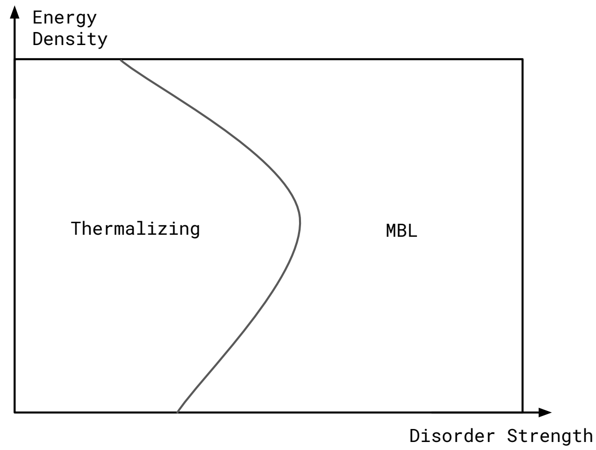

FIG. 1: Schematic phase diagram of disordered one dimensional spin chain with a MBL phase. Adopted from[11].

TABLE I: Existence of thermal and localized phases

	Anderson Localization	Many-Body Localization
$d = 1$	localized only	localized + thermal
$d = 2$	localized only	?
$d \geq 3$	localized + thermal	?

III. AVALANCHE ARGUMENT

Now let's examine the argument of[1] in detail. First, we notice that in a lattice with random site energies, it is possible to have rare regions in which all site energies are close to each other. These regions with atypically weak disorder can be locally thermalizing even if the global phase is MBL. We thus wish to study the evolution of a single thermal region in a MBL bulk phase. We imagine a spherical thermal bubble with radius ℓ_b coupled to localized spins with coupling strength exponentially decaying in distance to the bubble. A schematic of this is shown in Fig. 2. The key question is: does this small thermal bubble drive an avalanche that thermalize the entire MBL bulk, or does the MBL bulk ultimately suppress the influence of the thermal bubble?

The combined system is described by the Hamiltonian

$$H = H_b + H_s + H_{sb} \quad (6)$$

where H_b acts on the thermal bubble, H_s acts on the localized spins, and H_{sb} couples the bubble to the localized spins via some local operator.

The thermal bubble is assumed to obey ETH. If we denote the eigenstates and eigenvectors of H_b by $|b\rangle$ and $E(b)$, then the matrix elements of local operator V are [c.f. Eq. (2)]

$$V_{bb'} = \langle b|V|b'\rangle = \delta_{bb'} \langle V\rangle_\epsilon + \frac{1}{\sqrt{\rho}} \sqrt{v(\omega)} \eta_{b,b'} \quad (7)$$

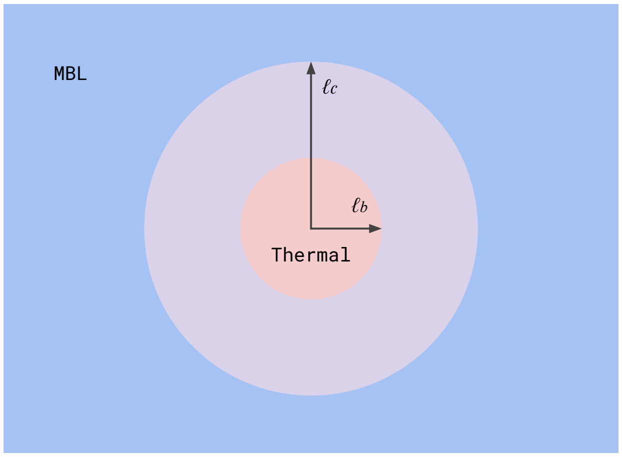

FIG. 2: A rare thermal bubble in MBL bulk phase. The bubble has radius ℓ_b. A hypothetical crossover region consisting of LIOMs well hybridized with the bubble extend out distance ℓ_c away from the bubble. Subsequent arguments show that such a crossover region is fully thermal and grows without bound.

where $\langle\cdot\rangle_\epsilon$ denote taking the thermal average at energy density ϵ, ρ is the density of states, $v(\omega)$ is a smooth, positive function of $\omega = E(b) - E(b')$, and $\eta_{b,b'}$ is a random variable of with zero mean and unit variance. Implicitly, we have assumed that the bubble ensemble is near energy density ϵ corresponding to maximum entropy, or infinite effective temperature. This is equivalent to assuming that H_b is a random matrix. $v(\omega)$ is the spectral function for operator V. In a bubble of linear dimension ℓ_b, there are approximately $2^{\ell_b^d}$ states, so the density of states (neglecting a volume normalization factor) is $\rho \approx 2^{\ell_b^d}/\mathcal{W}$, where $\mathcal{W}$ is the bubble bandwidth. The energy spacing between adjacent bubble energy eigenstates is then $1/\rho$.

The localized spins are described by LIOMs [c.f. Eq. (5)]

$$H_s = \sum_i h_i \tau_i^z \tag{8}$$

which do not interact with each other, and $|h_i| \leq W/2$. W is thus a "single particle" bandwidth for the localized system. Since interactions tend to destabilize locality, non-interacting LIOMs represent the best-case scenario for lozalization. The LIOMs are exponentially localized with localization length ζ.

LIOMs distance r away from the thermal region thus interact with the bubble via an exponentially decaying interaction with the same localization length:

$$H_{sb} = \sum_{i\in s} g_r V \otimes \tau_i^x, \quad g_r = g\exp[-r/\zeta] \tag{9}$$

First, we consider a LIOM immediately adjacent to the thermal bubble. The interaction term Eq. (9) tries to flip the localized spin, causing the thermal bubble to exchange energy $2h$ with the localized spin. The relevant matrix element in perturbation theory is $g\sqrt{v(2h)/\rho}$. If

$$g\sqrt{v(2h)/\rho} \gg 1/\rho \tag{10}$$

then we say the LIOM is absorbed into the the bubble. The new combined system has twice the Hilbert space dimension and half the level spacing. The authors show that under certain assumptions, the new eigenstates are random superpositions of old eigenstates, and the new system stil obeys ETH with a new spectral function $v'(\omega)$ almost identical to the original $v(\omega)$.

The initial condition Eq. (10) will be satisfied for large enough initial bubble size ℓ_b. Then, we can iterate this procedure so that the bubble-LIOM combined thermal system extends out to a larger radius $\ell_c + \ell_b$. The density of states is now $\rho' \approx 2^{(\ell_c+\ell_b)^d}/\mathcal{W}$. The matrix element for interacting with a new LIOM distance ℓ_c from the original bubble boundary is $g\exp[-\ell_c/\zeta]\sqrt{v(2h)/\rho'}$. The condition for the new LIOM to become well-hybridized with the thermal system is

$$g\sqrt{v(2h)}\exp\left[\frac{1}{2}(\ell_c+\ell_b)^d - \ell_c/\zeta\right] \gg 1 \tag{11}$$

The condition Eq. (11) can always be satisfied for large enough ℓ_c in dimensions $d > 1$, so the thermal system continues to grow. Intuitively, the superexponential growth of the thermal Hilbert space overwhelms the exponentially decaying interactions. From this inductive scenario, the authors conclude that MBL is unstable in dimensions $d > 1$, because any large enough rare thermal bubble will create an avalanche that delocalizes the entire system.

IV. DISCUSSION

The argument[1] presented above has been tested numerically in small systems[12] with good agreement. It is also consistent with the existence of MBL in one dimension, only placing an upper bound on the localization length ζ[13]. Personally, I find this argument convincing, especially since there has been no other rigorous theoretical work suggesting that MBL is stable in higher dimensions.

However, subsequent works have also presented valid critiques. The argument of[1] relies on the strong assumptions that the thermal region is modeled by random matrix theory throughout its growth. Random matrix theory ignores inevitable correlations in the bubble Hamiltonian. It is also not physical to assume the spectral function $v(\omega)$ remains entirely unchanged as new LIOMs are incorporated into the thermal system. We should expect some "back-action" from the MBL system on the bubble which modify the bubble spectral function and level statistics.[12] suggests the back-action may require the initial bubble size to be larger than estimated by[1] for the avalanche to succeed.

Furthermore, even if the argument[1] is essentially valid, we should keep in mind that thermal bubbles are exponentially rare events. The system size required for a large enough thermal bubble to appear with reasonable probability may soon exceed length scales accessible in experiment. The timescales required for the core bubble to hybridize with distant LIOMs is also exponentially long. Thus, for strongly localized systems with experimentally accessible length- and time-scales, the system can act localized, even if they are ultimately thermal in the asymptotic limit. This may be what is occuring in experiments that claim to observe signatures of MBL in two dimensions[2].

We have not yet reached the the end of the road for avoiding thermalization in higher dimensions either. MBL may still be stable in quasiperiodic systems, where rare thermal regions cannot occur[14,15]. People are also searching for other generic mechanisms for ergodicity-breaking, such as Hilbert space shattering[16] and Stark localization[17].

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